

Duals of L^p -spaces
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All vector spaces in this document are over the field \mathbb{R} of real numbers.

Chapter 1

Mise en bouche

Theorem 1.1. *The usual topology on \mathbb{R}^n is the only Hausdorff topology that makes it a topological vector space.*

Proof. Let \mathcal{T} be a Hausdorff topology on \mathbb{R}^n making it a topological vector space. The identity map $\text{id}: (\mathbb{R}^n)_{\text{can}} \rightarrow (\mathbb{R}^n)_{\mathcal{T}}$, where the first space is endowed with the usual topology and the second with \mathcal{T} is a continuous (linear) bijection between Hausdorff spaces, and it is therefore a homeomorphism.

Of course, the whole point is to state this in Lean. □

Theorem 1.2. *The only topological vector space over \mathbb{R} with the discrete topology is the zero space.*

Proof. Let E be a discrete, real, topological vector space: for every $v \in E$, by continuity of the multiplication, the line $\mathbb{R} \cdot v$ is both connected and discrete, so it is a singleton. This shows $v = 0$. □

Chapter 2

Locally Convex and Bolognese Spaces

In this chapter, all Banach spaces are supposed to be *non-zero*. Observe that the definition `BanachSpace` does not exist in Mathlib: before starting off, make sure that you know how to state that a space V is Banach.

2.1 Duals of Banach Spaces

Theorem 2.1. *If V is a Banach space and $v \neq 0$, then some $\varphi \in V^*$ satisfies $\varphi(v) \neq 0$.*

Proof. The result is in Mathlib *verbatim*, the problem is to find it.

Suggestion : prove first the **separation result 2.2**. □

Theorem 2.2. *Let V be a Banach space and let $v \neq w$ be two distinct elements of V . There is a $\varphi \in V^*$ such that $\varphi(v) \neq \varphi(w)$.*

Proof. See the proof of Theorem 2.1. *Extra questions:* Are you sure that completeness is necessary? □

2.2 Bolognese Spaces

Definition 2.3. A topological vector space is Bolognese if the convex open sets are a base for the topology: for every open set U around a point, there is a convex open set C containing that point such that $C \subseteq U$.

The above definition is *almost identical* to the definition of **Locally Convex Space** of Mathlib.

Theorem 2.4. *A topological vector space is Bolognese if and only if it is Locally Convex.*

Proof. The proof consists in unfolding the definition of a basis of a topological space, using that when V is a topological vector space (so, addition and scalar multiplications are continuous), there is a basis of the neighborhoods of 0 made of open, absolutely convex spaces (this is in Mathlib). □

Theorem 2.5. *Every Banach space is Bolognese.*

Proof. Since Banach spaces are locally convex, this follows from Theorem 2.4. On the other hand, completeness is probably useless: try to generalise. \square

Theorem 2.6. *For every $1 \leq p$ the space $L^p([0, 1])$ (where $[0, 1]$ is endowed with the restriction of the Lebesgue measure) is Bolognese.*

Proof. All the ingredients are in Mathlib: the only “problem” is how to state the theorem, namely how to speak about $L^p([0, 1])$. \square

Finally, we see that Theorem 2.2 was a special case of a more general result:

Theorem 2.7. *Let V be a Hausdorff Bolognese space and let $v \neq w \in V$ be distinct elements of V . There is a $\varphi \in V^*$ such that $\varphi(v) \neq \varphi(w)$.*

Proof. Once more, the proof is basically a matter of finding the right formulation and right statement in Mathlib. Observe (*i. e.*: try!) that you can assume T1 instead of Hausdorff, if you so wish. \square

2.3 Non-Bolognese Spaces

It comes the question of understanding whether every space is Bolognese and, if not, to produce examples of non-Bolognese ones.

Theorem 2.8 (Tychonoff (1935), see [Tyc35]). *For every $0 < p < 1$, the space ℓ^p of sequences $(x_n)_{n \in \mathbb{N}}$ of real numbers such that $\sum_{n \geq 0} |x_n|^p < \infty$ is not Bolognese.*

Proof. Arguing by contradiction, suppose that the unit ball $\mathcal{B}(0, 1) \subseteq \ell^p$ contains a convex open $0 \ni U \subseteq \mathcal{B}(0, 1)$: then U contains a ball $\mathcal{B}(0, \varepsilon)$ for some ε .

For each $1 \leq i \leq \infty$, let $x_i = (\varepsilon \delta_{i,j})_j$ be the vector whose coordinates are 0 at all $j \neq i$ and whose i -th coordinate is ε : clearly, $x_i \in \mathcal{B}(0, \varepsilon) \subseteq \ell^p$. Consider now, for each $N \in \mathbb{N}$, the vector

$$y_N = \sum_{i=1}^N \frac{1}{C_N \cdot i^{1/p}} x_i = (0, \dots, \frac{\varepsilon}{C_N i^{1/p}})$$

where $C_N = \sum_{i=1}^N i^{1/p}$. Clearly all y_N are (finite) linear combinations of the x_i 's, so they all belong to the convex set $U \subseteq \mathcal{B}(0, 1)$. On the other hand,

$$\|y_N\| = \sqrt[p]{\left(\sum_{i=1}^N \frac{\varepsilon}{C_N \cdot i^{1/p}}\right)^p} = \sqrt[p]{\sum_{i=1}^N \left|\frac{\varepsilon^p}{C_N^p \cdot i}\right|} = \frac{\varepsilon}{C_N} \sqrt[p]{\sum_{i=1}^N \frac{1}{i}}.$$

Now, since

$$\lim_{N \rightarrow +\infty} \frac{1}{\sum_{i=1}^N \sqrt[p]{i}} \sqrt[p]{\sum_{i=1}^N \frac{1}{i}} = \lim_{N \rightarrow +\infty} \frac{1}{\sum_{i=1}^N \sqrt[p]{i}} \sqrt[p]{\sum_{i=1}^N \frac{1}{i}} = +\infty$$

there exists N such that $\|y_N\| > 1$, contradicting the assumption that $y_N \in \mathcal{B}(0, 1)$ for all N . \square

Passing from numerical sequences to measurable functions yields other examples:

Theorem 2.9. *For every $0 < p < 1$, the space $L^p([0, 1])$ of equivalence classes of measurable functions $f: [0, 1] \rightarrow \mathbb{R}$ satisfying $\int_0^1 |f(x)|^p dx < \infty$ (with respect to the Lebesgue measure) is not Bolognese.*

Proof. Combine Theorem [2.10](#) and Theorem [2.7](#). □

The above theorem can be proven directly, but we rather deduce it as a corollary of the following one, combined with Theorem [2.7](#):

Theorem 2.10. *For every $0 < p < 1$ the dual space $L^p([0, 1])^*$ is equal to 0.*

Proof. For a direct proof, see [\[Con\]](#). □

Chapter 3

Other Non-Bolognese Spaces

The results in this chapter are all borrowed from [Con] and none of them has been formalised during the workshop.

Theorem 3.1. *Let (X, μ) be a measure space. For every $0 < p$ the space $L^p(\mu)$ is a metric space with metric*

$$d(f, g) = \begin{cases} \left(\int_X |f - g|^p \, d\mu \right)^{1/p}, & \text{if } p \geq 1 \\ \int_X |f - g|^p \, d\mu, & \text{if } 0 < p < 1 \end{cases}$$

Can Theorem 2.9 be generalised beyond the case $X = [0, 1]$? The answer relies on the following

Theorem 3.2. *For every $0 < p < 1$, the space $L^p(\mu)$ is Bolognese if and only if μ assumes finitely many values.*

As a corollary of the above result, we obtain a new proof of Theorem 2.9:

Corollary 3.3. *The space $L^p([0, 1])$ of equivalence classes of measurable functions $f: [0, 1] \rightarrow \mathbb{R}$ satisfying $\int_0^1 |f(x)|^p dx < \infty$ (with respect to the Lebesgue measure) is not Bolognese.*

Proof.

□

Bibliography

- [Con] K. Conrad. L^p -spaces for $0 < p < 1$. URL: <https://kconrad.math.uconn.edu/blurbs/analysis/lpspace.pdf> (cited on pages 5, 6).
- [Tyc35] A. Tychonoff. Ein Fixpunktsatz. German. *Math. Ann.*, 111:767–776, 1935. DOI: [10.1007/BF01472256](https://doi.org/10.1007/BF01472256). URL: <https://eudml.org/doc/159810> (cited on page 4).